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SYNTHETIC ASPECTS OF C^∞-MAPPINGS

Marta BUNGE
Department of Mathematics, McGill University, and Groupe Interuniversitaire en Études Catégoriques Montréal, P. Québec, Canada

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0. Introduction

This paper deals with some aspects of the theory of C^∞-mappings which may be profitably studied in the context of Synthetic Differential Geometry.

A special model that is particularly adapted to test our synthetic definitions and statements is the topos $\mathbf{B}^{\mathrm{op}}$ described by E. Dubuc (cf. [4], also [15], and [9]). The category $\mathbf{B}$ is the category of C^∞-rings of the form $C^\infty(\mathbf{R}^n)/I$, where I is a local ideal. This means that, for $f\in C^\infty(\mathbf{R}^n)$, $f\in I$ if and only if the germ $f\,|_x$ belongs to $I\,|_x$, the ideal generated by the germs $g\,|_x$ of $g\in I$. The Grothendieck topology on $\mathbf{B}^{\mathrm{op}}$ is generated by the families of the form $(C^\infty(U)\to C^\infty(U_j))_{j\in J}$, where U is open in $\mathbf{R}^n$ and $\{U_j\}_{j\in J}$ is an open covering of U.

The starting point is the synthetic theory of jets given by A. Kock in [8], of which we develop the analogue for germs (Section 1). This is based on a synthetic notion of germ which is implicit in the work of J. Penon in [11]. A suggestion of A. Kock (in [9], p. 285) then led us to a synthetic formulation and proof of the Preimage theorem (Section 2). In order to prove some density results, some version of Sard's theorem is needed. We give one that is true in $\mathbf{B}^{\mathrm{op}}$, and then use it to give a proof that a certain class of immersions is 'dense'. We then study transversality from the synthetic point of view, and establish, synthetically, two results: the Transversal Preimage theorem and Thom's transversality theorem (Section 4). An application of the latter is the density of Morse functions (Section 5). The basic notion of stability of smooth mappings is investigated in one of its forms, namely, infinitesimal stability, as this is the most natural from the point of view of Synthetic Differential Geometry (Section 5). Finally, we just hint at the advantages of studying unfoldings of singularities from the synthetic point of view (Section 6).

We assume familiarity with A. Kock's textbook and reference work [9]. Throughout the sections of this paper, assumptions about the ring object R in the general topos $\mathscr{E}$ in which we work, are made in the measure that they become needed. These assumptions are intended to be cumulative, but nowhere in this paper are they listed together. They are all valid in $\mathbf{B}^{\mathrm{op}}$.

1. Synthetic theory of germs

Let $\mathscr{E}$ be a topos, and R a non-trivial commutative ring in $\mathscr{E}$ assumed to be *of line type* (or to satisfy the *Kock–Lawvere axiom*) in the strong sense (cf. [9], p. 28). Explicitly, for each $n \geq 1$, $k \geq 1$, denote by

$$D_k(n) = [\![y = (y_1, \dots, y_n) \in R^n \mid \text{any product of } k+1 \text{ or more of the } y_1, \dots, y_n \text{ is } 0]\!],$$

and for $n \geq 1$, $k \geq 1$, $p \geq 1$, denote by $\sigma^{p,k,n} : R^s \to R^{p^{D_k(n)}}$ (where $s = p \cdot \binom{n+k}{k}$), the map whose exponential adjoint $R^s \times D_k(n) \to R^p$ is given by the rule: $((a_{\alpha i}), x) \to (\sum_{\alpha \leq k} a_{\alpha i} x^\alpha)$, where $1 \leq \alpha \leq \binom{n+k}{k}$ and $1 \leq i \leq p$. With these notations, the axiom may be stated:

For each $n \geq 1$, $k \geq 1$, $p \geq 1$, the map $\sigma^{p,k,n}$ is an isomorphism.

The infinitesimal subobjects $D_k(n) \hookrightarrow R^n$, as well as their union $D_\infty(n) \hookrightarrow R^n$, play an essential rôle in Kock's synthetic theory of jets (cf. [8]). For $x \in R^n$, *a k-jet at x 'of a map from R^n to R^p'* is simply a map $x + D_k(n) \to R^p$ (not an equivalence class of maps). Maps $x + D_\infty(n) \to R^p$ are said to be *jets at x*. Notice that, in general, $x \in R^n$ is a generalized element defined, e.g., at stage A, where A is an object of $\mathscr{E}$. Then, $x + D_k(n) = [\![z \in R^n \mid z = x + y, \text{ for some } y \in D_k(n)]\!]$ makes sense as a subobject of R^n only in the topos $\mathscr{E}/A$, where by abuse of notation we still denote by R^n the image of R^n under the (logical) functor $\mathscr{E} \to \mathscr{E}/A$ given by $X \to \bar{X} = (X \times A \xrightarrow{\pi_2} A)$.

In order to deal with germs in an analogous way, one resorts to the infinitesimal neighborhoods of 0 considered by Penon [11]. These are given, for each $n \leq 1$, by the objects

$$\Delta(n) = \left[\!\!\left[y = (y_1, \dots, y_n) \in R^n \mid \neg\neg \left(\bigwedge_{i=1}^{n} y_i = 0 \right) \right]\!\!\right] = \neg\neg\{0\} \hookrightarrow R^n.$$

For $n = 1$, $\Delta(n)$ is denoted by Δ. In the model $\mathbf{B}^{\mathrm{op}}$, Δ is represented by the germs at 0 of smooth real valued functions, i.e., by $C_0^\infty(\mathbf{R})$ (cf. [11]). For $x \in R^n$, let us say that a map $x + \Delta(n) \to R^p$ is *a germ at x 'of a map from R^n to R^p'*. As before, if $x \in_A R^n$, then the above is to be interpreted appropriately as a map in $\mathscr{E}/A$.

One immediately notices that

$$x + \Delta(n) = \neg\neg\{x\} \tag{1.1}$$

and therefore, that the notion of germ that we have given is that which is contained in Penon's formulation (cf. [11]) of the Implicit Function theorem.

Let us assume next that R is a *field object* in the sense of Kock ([7], or [9] p. 267).

This means that, in addition to being as non-trivial commutative ring in $\mathscr{E}$, R satisfies the axiom

$$\forall x_1, \ldots, x_n \in R \quad \left[\neg \left(\bigwedge_{i=1}^{n} x_i = 0\right\} \to \bigvee_{i=1}^{n} (x_i \# 0)\right]$$

where "$x \# 0$" stands for "x is invertible". It follows that R is a *local ring*, in the sense that:

$$\forall x, y \in R \quad [(x+y) \# 0 \to (x \# 0) \vee (y \# 0)].$$

Also in view of the field axiom, the statements "$\neg(x=0)$" and "$(x \# 0)$" become interchangeable.

We may now state some simple facts:

$$\Delta(n) = \Delta^n. \tag{1.2}$$

This follows from the fact that $\neg\neg$ commutes with $\wedge$.

(1.3) $\Delta(n)$ is an R-linear subspace of R^n.

Use that R is a field.

(1.4) For any $x \in R^n$, there is an isomorphism $\Delta(n) \to x + \Delta(n)$, given by addition with x.

The synthetic basis for relating germs and jets is the inclusion, for each $n \geq 1$,

$$D_\infty(n) \hookrightarrow \Delta(n). \tag{1.5}$$

If $y = (y_1, \ldots, y_n) \in D_\infty(n)$, then for some $k \leq 1$, $y \in D_k(n)$ hence $\bigwedge_{i=1}^n (y_i^{k+1} = 0)$. This implies that $\bigwedge_{i=1}^n \neg(y_i^{k+1} \# 0)$ and so, that $\neg \bigvee_{i=1}^n (y_i \# 0)$, hence $\neg\neg(\bigwedge_{i=1}^n y_i = 0)$ so that $y \in \Delta(n)$.

Composition with (1.5) induces, for any $p \geq 1$, a map denoted

$$j_0^\infty : R^{p^{\Delta(n)}} \to R^{p^{D_\infty(n)}}$$

which is said to assign, to a germ at $0, \phi \in R^{p^{\Delta(n)}}$, its jet at 0. Similarly one defines j_x^∞ as well as j_x^k, for arbitrary $x \in R^n$, $k \geq 1$.

Composition with the inclusion $D_\infty(n) \hookrightarrow R^n$ induces a map also denoted by j_0^∞. The factorization of the above subobject through $\Delta(n) \hookrightarrow R^n$ (given by (1.5)) is a simple way to express the useful fact that the jet at a point of a map from R^n to R^p, depends solely on the germ at the point of the map. Similar statements can be made about jets and germs.

Closely related to the jet maps j_0^k are the following, which we will need in the proof of Thom's transversality theorem. For each $h \in R^{p^{R^n}}$ and $k \geq 0$, define

$$J^k h : R^n \to R^{p^{D_k(n)}}$$

as follows: for $x \in R^n$, let $J_h^k(x) = j_0^k h_x$ where $h_x \in R^{p^{R^n}}$ is the image of h under the map

$$R^{p^{\alpha_x}} : R^{p^{R^n}} \to R^{p^{R^n}}$$

induced by composition with $\alpha_x = x + (-) : R^n \to R^n$. If $x \in_A R^n$, h_x is to be interpreted in $\mathcal{E}/A$, as usual.

(1.6) Let $x \in R^n$ and let $x' \in \neg\neg\{x\}$. If $f : \neg\neg\{x\} \to R^p$, then $f(x') \in \neg\neg\{f(x)\}$.

This holds on account of the monotonicity of $\neg\neg$.
An immediate consequence is the useful remark that:

(1.7) Every map $\neg\neg\{x\} \to R^p$ taking x to y, factors through $\neg\neg\{y\} \hookrightarrow R^p$.

This has a number of consequences in the form of simplified definitions. Firstly, since a germ is a map $\neg\neg\{x\} \to \neg\neg\{y\}$ for some x, y, *composition of germs* is easy to define. Secondly, a germ $\phi : \neg\neg\{x\} \to \neg\neg\{x\}$ is an *invertible germ* if the map ϕ is an isomorphism. Finally, to say that germs $\phi : \neg\neg\{x\} \to \neg\neg\{y\}$ and $\phi' : \neg\neg\{x'\} \to \neg\neg\{y'\}$ are *equivalent germs*, all we have to say is that there exist invertible germs $\alpha : \neg\neg\{x\} \to \neg\neg\{x'\}$ and $\beta : \neg\neg\{y\} \to \neg\neg\{y'\}$ such that $\phi' = \beta \circ \phi \circ \alpha^{-1}$. Write $\phi \sim \phi'$ in this case. These definitions are to be interpreted in an *internal* sense.

Restriction to $\neg\neg\{x\} \to R^n$ for $x \in R^n$ induces for each $p \geq 1$ a map denoted

$$R^{p^{R^n}} \xrightarrow{|_x} R^{p^{\neg\neg\{x\}}}$$

and is said to assign, to a map $f \in R^{p^{R^n}}$, its germ $f|_x$ at x.

In [8], the notion of manifold introduced is that of a formal manifold and fits well in the context of jets or formal power series. When dealing with germs, it is another notion of manifold that is appropriate, and is that of Penon [11], which is stronger. We shall use it in the following form here:

Let $M \to R^n$. Call M a *submanifold* of R^n of *dimension* $r \leq n$ (or of *codimension* $n-r$), if for each $x \in R^n$ there is given an isomorphism $\alpha : \neg\neg\{x\} \to \neg\neg\{0\}^n$ such that the restriction of α to $\neg\neg\{x\} \cap \hookrightarrow \neg\neg\{x\}$ maps $\neg\neg\{x\} \cap M$ onto $\neg\neg\{0\}^r \hookrightarrow \neg\neg\{0\}^n$ (this inclusion given by $(x_1, \ldots, x_r) \mapsto (x_1, \ldots, x_r, 0, \ldots, 0)$ and denoted by u_r^n). This definition is to be interpreted in an internal sense.

We shall have occasion to use this definition in stating the Preimage theorem in the next section. When so doing, however, it will be necessary to strengthen this notion even further; we shall speak of *manifolds 'cut out by independent functions'*.

2. Regular values

As pointed out by Kock (in [9], p. 285), a synthetic formulation (and proof) of the Preimage theorem (cf. [6], p. 21) is still lacking. We give them in this section.

Denote by $D : R^{p^{R^n}} \to R^{p n^{R^n}}$ the *Jacobian map*, easily constructed synthetically by means of the partial derivative operators $\partial/\partial x_i$ (cf. [9], p. 55). If $x \in R^n$ and $f \in R^{p^{R^n}}$, $D_x f = (\partial f_j(x)/\partial x_i)_{ji}$, where $f = (f_1, \ldots, f_p)$. Closely related is the R-linear map

$$df_x : T_x R^n \to T_{f(x)} R^p,$$

induced from x, f, by forming the induced map between tangent bundles in the synthetic context, i.e.,

$$f^D : R^{n^D} \to R^{p^D}$$

and then restricting to the fiber of $R^{n^D} \xrightarrow{\pi_0} R^n$ above x. As R-linear spaces $T_x R^n \approx R^n$ and $T_{f(x)} R^p \approx R^p$, and modulo these isomorphisms df_x is represented by $D_x f$ in the canonical bases. In particular, df_x is epimorphic if and only if $D_x f$ has a right inverse.

Some basic Linear Algebra is needed in order to proceed from here.

We say that an n-tuple of elements $y_1, \ldots, y_n \in R^p$ forms a *linearly independent set* if

$$\forall \lambda_1, \ldots, \lambda_n \in R \left[\bigvee_{i=1}^{n} \lambda_i \# 0 \to \sum_{i=1}^{n} \lambda_i y_i \# 0 \right].$$

It is shown in [17] that, over a field, this notion is equivalent to the notion of linear independence which occurs in [7]. In particular, the following two results hold:

(2.1) For any $p \times n$ matrix $X \in R^{pn}$, row Rank $X \geq r$ iff column Rank $X \geq r$.

(2.2) Let $X \in R^{pn}$ and assume Rank $X = p$. Then, locally, X has a right inverse. (Recall that for $r = p$ or n, one writes Rank $X = r$ for row Rank $X \geq r$ or column Rank $X \geq r$).

The key notion in this section is that of a submersion. If $f \in R^{p^{R^n}}$ and $x \in R^n$, say that f is *a submersion at* x if Rank $D_x f = p$. (Notice that this implies that $n \geq p$.) Call f a *submersion* if f is a submersion at x for every $x \in R^n$.

From (2.1) and (2.2) above we deduce the following useful equivalent descriptions of a submersion.

2.3. Proposition. *Let $f \in R^{p^{R^n}}$, $x \in R^n$. Then, the following are equivalent:*

(i) *f is a submersion at x.*

(ii) $\bigvee_{(i_1, \ldots, i_p) \in \binom{n}{p}} \{\partial f(x)/\partial x_{i_1}, \ldots, \partial f(x)/\partial x_{i_p}\}$ *is linearly independent.*

(iii) *df_x is locally surjective.*

(What (iii) means is that the statement "$\forall v \in R^{p^D} [\pi_0 v = fx \to \exists u \in R^{n^D} (\pi_0 u = x \wedge f^D u = v)]$" holds in the topos $\mathcal{E}$).

At this point, we make the assumption about R that the *Inverse Function*

Theorem (cf. [11]) holds. (Recall that it does hold in our test model $\mathbf{B}^{\mathrm{op}}$, as shown in [11].) We state it in the following way:

$$\forall \phi \in R^{n^{\Delta(n)}}\ [(\phi(0)=0 \to j_0^1\phi \text{ iso}) \to \phi \text{ iso}].$$

The theorem below is usually obtained as a special case of the Rank theorem (cf. [3], p. 2) and, unlike the Rank theorem itself, it is easy to show in our context and is sufficient for proving the Preimage theorem.

2.4. Theorem (Submersion theorem). *Let $f \in R^{p^{R^n}}$, $x \in R^n$ with f a submersion at x. Then $f|_x$ is locally equivalent to $\pi^n_p|_0$, where $\pi^n_p : R^n \to R^p$ is the projection described by $(x_1, \ldots, x_p, \ldots, x_n) \mapsto (x_1, \ldots, x_p)$.*

Proof. Because of (1.4) we can restrict ourselves to the case $x=0$ and $fx=0$. That is, instead of dealing with $f|_x$ we shall take the composite

$$\neg\neg\{0\}^n \xrightarrow{\sim} \neg\neg\{x\} \xrightarrow{f|_x} \neg\neg\{fx\} \xrightarrow{\sim} \neg\neg\{0\}^p$$

instead. First, the local right invertibility of the Jacobian at x (which depends solely on the germ at x) is not affected. Secondly, if the above composite ends up being locally equivalent to $\pi^n_p|_0$, so will $f|_x$ itself.

Assume that f is defined at stage $A \in \mathscr{E}$. Notice that $f|_0 : \neg\neg\{0\}^n \to \neg\neg\{0\}^p$ is a map in $\mathscr{E}/A$ and thus we must consider $\pi^n_p|_0$ also in $\mathscr{E}/A$. Since f is a submersion at 0, by Proposition 2.3(ii) there is a jointly epimorphic family $\{A_i \xrightarrow{\zeta_i} A\}_{i\in I}$ in $\mathscr{E}$, such that for each $i \in I$ there is a p-tuple $(i_1, \ldots, i_p)$ such that $\{\zeta_i^*(\partial f(0)/\partial x_{i_1}, \ldots, \zeta_i^*(\partial f(0)/\partial x_{i_p})\}$ is linearly independent (in $\mathscr{E}/A_i$). The uniqueness is the axiom of line type allows us to translate the above into $\{\partial(\zeta_i^* f)(0)/\partial x_{i_1}, \ldots, \partial(\zeta_i^* f)(0)/\partial x_{i_p}\}$ linearly independent (in $\mathscr{E}/A_i$).

What we need to show is that, for each $i \in I$, there is a jointly epimorphic family $\{B_{ij} \xrightarrow{\gamma_{ij}} A_i\}_{j\in J_i}$ such that, for each $j \in J_i$, one has $\gamma_{ij}^*(\zeta_i^* f)|_0 \sim \pi^n_p|_0$ (in $\mathscr{E}/B_{ij}$). Composing coverings will give a covering $\{B_k \xrightarrow{\zeta_k} A\}_{k\in K}$ and the desired conclusion. The argument will therefore be the same were we to suppose that $\{\partial f(0)/\partial x_1, \ldots, \partial f(0)/\partial x_p\}$ are linearly independent in $\mathscr{E}/A$, which we now do, for the sake of simplicity.

Define $\phi \in_A R^{n^{R^n}}$ by $\phi = \langle f, \pi^n_{n-p}\rangle$. Clearly, $\phi(0)=0$ and the Jacobian of ϕ at 0 is given by the matrix

$$\begin{array}{cc}
\left[\begin{array}{c|c}
\multicolumn{2}{c}{\left(\dfrac{\partial f_j}{\partial x_i}(0)\right)_{ji}} \\
\hline
* & \begin{matrix} 1 & & & \\ & 1 & 0 & \\ & & 1 & \\ & 0 & & \\ & & & 1 \end{matrix}
\end{array}\right]
&
\begin{array}{l} \}\ p \\ \\ \\ \}\ n-p \\ \\ \end{array}
\\
\begin{array}{cc} \underbrace{\hphantom{xxxxxx}}_{p} & \underbrace{\hphantom{xxxxxx}}_{n-p} \end{array} &
\end{array}$$

By assumption, $\operatorname{Rank} D_0 f = p$. Thus, $\operatorname{Rank} D_0\phi = n$ and ϕ is a submersion at 0. By Proposition 2.3(iii) $d\phi_0$ is locally surjective, hence bijective (by the analogue of Ex. 10.1 in [9]). Hence, $j_0^1\phi$ is locally an isomorphism and so, by the Inverse Function theorem, $\phi\,|_0$ is also locally an isomorphism. By the uniqueness of inverses, it is enough to suppose that there is $B \xrightarrow{\gamma} A$ such that $\gamma^*(\phi\,|_0) = \gamma^*\phi\,|_0$ is an invertible germ in $\mathscr{E}/B$. Denote by g the composite

$$\neg\neg\{0\}^n \xrightarrow{(\gamma^*\phi\,|_0)^{-1}} \neg\neg\{0\}^n \xrightarrow{\gamma^* f\,|_0} \neg\neg\{0\}^p.$$

From the identity $\gamma^*\phi\,|_0 \circ g = \gamma^* f\,|_0$ follows that if $x = (x_1, \ldots, x_n) \in \neg\neg\{0\}^n$, then

$$\begin{tikzcd} x \arrow[r, mapsto, "\gamma^*\phi\,|_0"] \arrow[dr, mapsto, "\gamma^* f\,|_0"'] & (\gamma^* f_1(x), \ldots, \gamma^* f_p(x), x_{p+1}, \ldots, x_n) \arrow[d, mapsto, "g"] \\ & (\gamma^* f_1(x), \ldots, \gamma^* f_p(x)) \end{tikzcd}$$

and so, that g is $\pi_p^n\,|_0$. This says that $\gamma^* f\,|_0 \sim \pi_p^n\,|_0$ as required. $\square$

Let $f \in R^{p^{R^n}}$, $y \in R^p$ be given at the same stage. We say that y is *a critical value of f* provided

$$\exists x \in R^n \left[(f(x) = y) \wedge \bigwedge_{H \in \binom{n}{p}} \det(D_x f)_H = 0 \right],$$

and y is *a regular value of f* provided

$$\neg(y \text{ is a critical value of } f).$$

Equivalently, using the field property, y is a regular value of f iff

$$\forall x \in R^n \left[(f(x) \# y) \vee \bigvee_{H \in \binom{n}{p}} \det(D_x f)_H \# 0 \right],$$

iff

$$\forall x \in R^n\ [(f(x) \# y) \vee f \text{ is a submersion at } x].$$

2.5. Corollary (Preimage theorem). $\forall f \in R^{p^{R^n}}\ \forall y \in R^p$ [*y is a regular value of $f \to M = f^{-1}\{y\}$ is a submanifold of R^n of dimension* $(n-p)$].

Proof. Assume f, y to be given both at stage A. If $x \in_A M$, then $fx = y$ so that f is necessarily a submersion at x. By Theorem 2.4, $f\,|_x$ is locally equivalent to $\pi_p^n\,|_0$. Thus, there is a jointly epimorphic family $\{B_i \xrightarrow{\gamma_i} A\}_{i \in I}$ and for each $i \in I$, isomorphisms α_i, β_i so that

$$\begin{tikzcd} \gamma_i^*(\neg\neg\{x\}) \arrow[r, "\gamma_i^*(f\,|_x)"] \arrow[d, "\alpha_i"'] & \gamma^*(\neg\neg\{fx\}) \arrow[d, "\beta_i"] \\ \neg\neg\{0\}^n \arrow[r, "\pi_p^n\,|_0"'] & \neg\neg\{0\}^n \end{tikzcd}$$

commutes in $\mathscr{E}/B_i$. (We do not change the notation of the projections when passing from $\mathscr{E}$ to $\mathscr{E}/A$ or from $\mathscr{E}/A$ to $\mathscr{E}/B_i$ since these functors preserve products.) Consider now the following pullback diagram in $\mathscr{E}/A$:

$$\begin{tikzcd}
\neg\neg\{x\}\cap M \arrow[r] \arrow[d] & f^{-1}\{y\} \arrow[r] \arrow[d] & 1 \arrow[d, "y"] \\
\neg\neg\{x\} \arrow[r] \arrow[rr, bend right, "f|_x"'] & R^n \arrow[r, "f|_x"'] & R^p
\end{tikzcd}$$

It says that

$$(f|_x)^{-1}\{y\} \cong \neg\neg\{x\}\cap M.$$

To show: there is an isomorphism σ, making the diagram below commutative:

$$\begin{tikzcd}
\neg\neg\{0\}^{n-p} \arrow[r, "\sigma"] \arrow[rdd, hook] & \gamma_i^*(f|_x)^{-1}\{y\} \arrow[d, hook] \\
& \gamma_i^*(\neg\neg\{x\}) \arrow[d, "\alpha_i"] \\
& \neg\neg\{0\}^n
\end{tikzcd}$$

But this follows readily from consideration of the following pullback diagrams, the first because of γ_i^* is logical, the second because of (1.2):

$$\begin{tikzcd}
\gamma_i^*((f|_x)^{-1}\{y\}) \arrow[r] \arrow[d, hook] & 1 \arrow[d] \\
\gamma_i^*(\neg\neg\{x\}) \arrow[r] \arrow[d, "\alpha_i"'] & \gamma^*(\neg\neg\{y\}) \arrow[d, "\beta_i"] \\
\neg\neg\{0\}^n \arrow[r] & \neg\neg\{0\}^p
\end{tikzcd}$$

and

$$\begin{tikzcd}
\neg\neg\{0\}^{n-p} \arrow[r] \arrow[d, hook] & 1 \arrow[d, hook, "\ulcorner 0 \urcorner"] \\
\neg\neg\{0\}^n \arrow[r, "\pi_p^n|_0"'] & \neg\neg\{0\}^p
\end{tikzcd}$$

To finish the proof, observe now that $\beta_i \cdot \ulcorner y{*}y \urcorner = \ulcorner 0 \urcorner$. $\square$

3. Sard's theorem and density of a class of immersions

We wish to establish, in our context, several density results all of which follow, classically, from a theorem of Sard ([6], p. 39). In the formulation below (for which we assume, from now on, that R is partially ordered in the sense of [2]), we shall refer to it as *Sard's axiom* ('Γ' denotes 'global sections'):

$$\forall \varepsilon \in \Gamma(R)\ \forall f \in R^{p^{R^n}}\ [\varepsilon > 0 \to \neg\, \forall y \in (-\varepsilon, \varepsilon)^p : y \text{ critical value of } f].$$

The positive version of it (not intuitionistically equivalent to the above) will be referred to, instead, as *the axiom of density of regular values:*

$$\forall \varepsilon \in \Gamma(R)\ \forall f \in R^{p^{R^n}}\ [\varepsilon > 0 \to \exists y \in (-\varepsilon, \varepsilon)^p : y \text{ regular value of } f].$$

A full discussion of the validity of these axioms in $\mathbf{B}^{\mathrm{op}}$ will be postponed for a later paper. At present, we shall limit ourselves to the consideration of the *global* version of the latter, and its immediate consequences.

3.1. Theorem. *In* $\mathbf{B}^{\mathrm{op}}$, *the following is valid:*

$$\forall \varepsilon \in \Gamma(R)\ \forall f \in \Gamma(R^{p^{R^n}})\ [\varepsilon < 0 \to \exists y \in \Gamma((-\varepsilon, \varepsilon)^p) : y \textit{ regular value of } f].$$

Proof. Let $\varepsilon \in R$, $\varepsilon > 0$ and let $f \in C^\infty(\mathbf{R}^n, \mathbf{R}^p)$. By Sard's theorem there is $y \in \mathbf{R}^p$, $y \in (-\varepsilon, \varepsilon)^p$ with y a regular value of f. We claim that the statement: $\forall x \in R^n$ $[fx = y \to f$ is a submersion at $x]$ follows from the above. Let $A = (C^\infty(\mathbf{R}^r)/I)$ be an object of $\mathbf{B}$. For $x \in_A R^n$, x is represented, modulo I, by some $X \in C^\infty(\mathbf{R}^r, \mathbf{R}^n)$. The assumption $fx = y$ translates into the statement that for every $t \in Z(I)$, the zeros of the ideal I (cf. [4], [15], or [2]), $f(X(t)) = y$. Since y is a regular value of f, given $t \in Z(I)$ we must have p of the vectors in $\{\partial f(X(t)/\partial x_1), \ldots, \partial f(X(t)/\partial x_n)\}$ linearly independent, although they may not always be the same p vectors for each $t \in Z(I)$.

Let

$$U = U_{(i_1, \ldots, i_p)} = \left\{ t \in R^n \;\middle|\; \left\{ \frac{\partial f}{\partial x_{i_1}}(X(t), \ldots, \frac{\partial f}{\partial x_{i_p}}(X(t)) \right\} \text{ linearly independent} \right\}.$$

Observe that U is open in $\mathbf{R}^r$ since both X and $\partial f/\partial x_i$ as well as "determinant" are continuous functions, so that the assertion that a certain $p \times p$ minor is invertible for a given t, remains valid on some neighborhood of t in $\mathbf{R}^n$. Also, the $W = W_{(i_1, \ldots, i_p)} = (U_{(i_1, \ldots, i_p)} \cap Z(I))$ cover $Z(I)$ by the above remark. We now claim that

$$A_{U_{(i_1, \ldots, i_p)}} \models \left\{ \frac{\partial f}{\partial x_{i_1}}(x), \ldots, \frac{\partial f}{\partial x_{i_p}}(x) \right\} \text{ linearly independent}$$

where

$$A_{U_{(i_1,\dots,i_p)}} = (C^\infty(U_{(i_1,\dots,i_p)})/I \mid U_{(i_1,\dots,i_p)}).$$

Suppose then that there is given $\lambda_k \in_{A_U} R$ (denoting $U = U_{(i_1,\dots,i_p)}$) and that $A_U \models \bigvee_{k=1}^p \lambda_k \# 0$, so that, for some open covering $\{V_k\}_{k\in K}$ of U, $A_{V_k} \models \lambda_k \# 0$, for each $k \in K$. Then,

$$A_{V_k} \models \sum_{k=1}^p \lambda_k \cdot \frac{\partial f}{\partial x_k}(x) \# 0,$$

since $\lambda_k = \lfloor A_k \rfloor_{I \mid U}$ is invertible in V_k, so that the sum $\sum_{k=1}^p A_k(t) \cdot (\partial/\partial x_k)(X(t)) \neq 0$ for each $t \in V_k$. This finishes the proof, as the $W_{(i_1,\dots,i_p)}$ together with $\{t \in Z(I) \mid f(X(t)) \neq y\}$ constitute an open covering of $Z(I)$. $\square$

3.2. Remark. The axiom of density of regular values, stated only for maps $f: R^n \to R^p$, is easily extendible, in view of its local nature, to maps $f: M \to R^p$, where $M \hookrightarrow R^n$ is a submanifold in the sense of Section 2.

Let $k \geq 0$, $n \geq 1$ and $s = \binom{n+k}{n}$. The object of polynomials with coefficients from R, in n variables and of degree $\leq k$, is easily described as a subobject $P_k(n)$ of R^{R^n} which is isomorphic to R^s. Given $\varepsilon \in \Gamma(R)$, $\varepsilon > 0$, and $f \in P_k(n)$, *write* $|f| < \varepsilon$ to mean that, if $f(x) = \sum_{\alpha \leq k} a_\alpha \cdot x^\alpha$, $x = (x_1, \dots, x_n) \in R^n$, then for every $1 \leq \alpha \leq s$, $a_\alpha \in (-\varepsilon, \varepsilon)$. Let $P^\varepsilon_k(n) = [f \in P_k(n) \mid |f| < \varepsilon]$. Under the isomorphism $P_k(n) \approx R^s$ one has $P^\varepsilon_k(n) \approx (-\varepsilon, \varepsilon)^s$. From this and the canonical isomorphisms $R^s \xrightarrow{\sim} R^{p^{D_k(n)}}$, given by the line type assumption, follows that there is a natural way to talk about jets as being 'close to 0'. We will employ this notion directly, rather than introducing some version of the Whitney C^∞-topology (cf. [5], p. 42) which we don't really need in this work. However, some comments about the latter are made at the end of Section 5.

The first application of Sard's theorem concerns immersions. For $f \in R^{p^{R^n}}$ and $x \in R^n$, f is said to be *an immersion at* x if $\operatorname{Rank} D_x f = n$. We say that f is an *immersion* if f is an immersion at x for every $x \in R^n$.

3.3. Theorem (Density of immersions $R^n \to R^p$, $p \geq 2n$). *Assume $p \geq 2n$. Then the following holds:*

$$\forall \varepsilon \in \Gamma(R) \left[\varepsilon > 0 \to \forall h \in \Gamma(R^{p^{R^n}})\ \exists f \in \Gamma(R^{p^{R^n}}) \left(f = (f_1, \dots, f_p) \to \bigwedge_{i=1}^p f_i \in P^\varepsilon_1(n) \wedge (h+f) \textit{ is an immersion}\right)\right].$$

Proof. Let $\varepsilon \in \Gamma(R)$ with $\varepsilon > 0$. Let $h: R^n \to R^p$. Suppose (after rearranging) that $\{\partial h(x)/\partial x_1, \dots, \partial h(x)/\partial x_s\}$ is linearly independent for every $x \in R^n$, $1 \leq s \leq n$. If no such s exists, set $s = 0$ in this proof.

Define $\phi: R^{s+n} \to R^p$ by

$$\phi(\lambda, x) = \sum_{j=1}^s \lambda_j \frac{\partial h}{\partial x_j}(x) - \frac{\partial h}{\partial x_{s+1}}(x).$$

Notice that ϕ is also a global section. By the axiom of density of regular values, there is $a_{s+1} \in \Gamma(R^p)$ with $a_{s+1} \in (-\varepsilon, \varepsilon)^p$ and a_{s+1} a regular value of ϕ. Define $g_1 : R^n \to R^p$ by

$$g_1(x) = h(x) + a_{s+1} \cdot x_{s+1}.$$

By means of the rules for differentiation ([9], §I.2) we can prove that

$$\frac{\partial g_1}{\partial x_i}(x) = \frac{\partial h}{\partial x_i}(x) \quad \text{for every } x \in R^n,\ i \le s,$$

and

$$\frac{\partial g_1}{\partial x_{s+1}}(x) = \frac{\partial h}{\partial x_{s+1}}(x) + a_{s+1} \quad \text{for every } x \in R^n.$$

We now claim:

$$\left\{\frac{\partial g_1}{\partial x_1}(x), \dots, \frac{\partial g_1}{\partial x_s}(x), \frac{\partial g_1}{\partial x_{x+1}}(x)\right\} \text{ is linearly independent,}$$

for every $x \in R^n$.

It is clearly equivalent to try to show:

$$\bigwedge_{i=1}^{s} \left\{ \forall \lambda_1, \dots, \lambda_{i-1}, \lambda_{i+1}, \dots, \lambda_{s+1} \in R \right.$$
$$\left. \left[\forall x \in R^n \left(\sum_{j \ne i} \lambda_j \cdot \frac{\partial g_1}{\partial x_j}(x) - \frac{\partial g_1}{\partial x_j}(x) \right) \# 0 \right] \right\}. \tag{$*$}$$

For $i = s+1$, $(*)$ says:

$$\left(\sum_{j=1}^{s} \lambda_j \frac{\partial h}{\partial x_j}(x) - \frac{\partial h}{\partial x_{s+1}}(x) \right) - a_{s+1} \# 0. \tag{$*_1$}$$

Since $p \ge 2n$ and $s \le n$, $\phi : R^{s+n} \to R^p$ could never be a submersion at any $(\lambda, x) \in R^s \times R^n$. That is, one has $\forall (\lambda, x) \in R^s \times R^n\ \neg(\phi$ is a submersion at $(\lambda, x))$. From the assumption that a_{s+1} is a regular value of ϕ it follows that $\forall (\lambda, x) \in R^s \times R^n (\neg(\phi(\lambda, x)) = a_{s+1})$. Equivalently, $\forall (\lambda, x) \in R^s \times R^n\ (\phi(\lambda, x) - a_{s+1} \# 0)$. This establishes $(*_1)$ above.

For $i \le s$, $(*)$ says:

$$A = \sum_{\substack{j \le s \\ j \ne i}} \lambda_j \frac{\partial h}{\partial x_j}(x) + \lambda_{s+1} \frac{\partial h}{\partial x_{s+1}}(x) + \lambda_{s+1} a_{s+1} - \frac{\partial h}{\partial x_i}(x) \# 0.$$

Let

$$B = -\lambda_{s+1} \frac{\partial h}{\partial x_{s+1}}(x) - \lambda_{s+1} a_{s+1}.$$

Then,

$$A + B = \sum_{\substack{j \le s \\ j \ne i}} \lambda_j \frac{\partial h}{\partial x_j}(x) - \frac{\partial h}{\partial x_i}(x) \# 0$$

by assumption on h.

Since R is a local ring, either $A \# 0$ or $B \# 0$. Suppose that $B \# 0$. We claim that this implies that $A \# 0$, i.e., that $\neg(A=0)$, equivalently.

Indeed, assume $A=0$. This says:

$$\sum_{\substack{j \leq s \\ j \neq i}} \lambda_j \frac{\partial h}{\partial x_j}(x) - \frac{\partial h}{\partial x_i}(x) = -\lambda_{s+1}\left[\frac{\partial h}{\partial x_{s+1}}(x) + a_{s+1}\right].$$

Hence, multiplying both sides of this identity by $\lambda_{s+1} \# 0$, (since $B \# 0$) gives

$$\left(\sum_{\substack{j \leq s \\ j \neq i}} -(\lambda_j \lambda_{s+1}^{-1}) \frac{\partial h}{\partial x_j}(x) + \lambda_{s+1}^{-1} \frac{\partial h}{\partial x_i}(x) - \frac{\partial h}{\partial x_{s+1}}(x)\right) - a_{s+1} = 0,$$

contradicting that a_{s+1} is a regular value of ϕ. This proves our claim.

Repeat the procedure $n-(s+1)$ times after having obtained a_{s+1} and g_1 as above, thus getting $a_{s+1}, \ldots, a_n \in \Gamma(R^p)$ with $a_j \in (-\varepsilon, \varepsilon)^p$ for each $s+1 \leq j \leq n$, so that

$$g_n(x) = h(x) + a_{s+1}x_{s+1} + \cdots + a_n x_n$$

has all n partial derivatives forming a linearly independent set, hence g_n is a submersion.

Let $f(x) = a_{s+1}x_{n+1} + \cdots + a_n x_n$. Then, the above says that $(h+f)$ is a submersion and if $f=(f_1, \ldots, f_p)$ this gives $f_i \in P_1^{\varepsilon}(n)$, as required. □

4. Transversality

The notion of transversality (cf. [6], p. 27] is an extension of that of regular value. In order to state it in this context we need some prior notation.

For an R-module Y, if X_1 and X_2 are R-submodules, let us denote by $X_1 + X_2 = [\![x_1 + x_2 \mid x_1 \in X_1 \wedge x_2 \in X_2]\!]$. This is an R-submodule of Y.

Consider $f: R^n \to R^p$, $x \in R^n$. There is induced $df_x: T_x R^n \to T_{f(x)} R^p$ and the image, $\operatorname{Im}(df_x)$, is an R-submodule of $T_{f(x)} R^p$.

If $N \hookrightarrow R^p$ and $x \in R^n$ is such that $f(x) \in N$, then the induced map $T_{f(x)} N \to T_{f(x)} R^p$ is a linear monomorphism and identifies $T_{f(x)} N$ with an R-submodule of $T_{f(x)} R^p$ as well.

Now for the main definition in this section. Let $f: R^n \to R^p$, $x \in R^n$ and $N \hookrightarrow R^p$ so that $f(x) \in N$. Say that f is *transversal to N at x* (write $f \pitchfork_x N$) if $T_{f(x)} R^p = \operatorname{Im}(df_x) + T_{f(x)} N$. Say that f is *transversal to N* (and write $f \pitchfork N$) if

$$\forall x \in R^n \; [\neg(f(x) \in N) \vee f \pitchfork_x N].$$

4.1. Remark. If $N = \{y\}$ for some $y \in R^p$, then $f \pitchfork \{y\}$ iff y is a regular value of f. In this case, $T_y\{y\} = 0$ and $\operatorname{Im}(df_x) \cong T_{f(x)} R^p$ iff df_x is surjective.

4.2. Remark. If $f: R^n \to R^p$ is a submersion and $N \hookrightarrow R^p$ any, then $f \pitchfork N$.

We now need to introduce a notion. For $g_1, \dots, g_l : R^p \to R$, let

$$Z(g_1, \dots, g_l) = \bigcap_{j=1}^{l} g_j^{-1}\{0\}.$$

We say that $g_1, \dots, g_l$ are *'independent functions'* if:

$$\forall y \in Z(g_1, \dots, g_l)\ \forall u \in T_y R^p\ \ [\{(dg_1)_y(u), \dots, (dg_l)_y(u)\}$$
$$\text{is linearly independent}].$$

4.3. Remark. If $g_1, \dots, g_l$ are independent functions then $N = Z(g_1, \dots, g_l) \hookrightarrow R^p$ is a submanifold. This is because $g = (g_1, \dots, g_l): R^p \to R^l$ is a submersion and $N = g^{-1}\{0\}$. Apply the Preimage theorem (Corollary 2.5). In this case, we say that N is (a submanifold) *'cut out by independent functions'*, explicitly, we say this when the following statement holds:

$$\exists g_1, \dots, g_l \in R^{R^p}\ [N = Z(g_1, \dots, g_l) \wedge g_1, \dots, g_l \text{ independent functions}].$$

4.4. Remark. Although this is a local definition (in the sense of [9], p. 175), the notion of submanifold being local as well, it follows that N *is* a submanifold of R^p.

The following generalizes the Preimage theorem:

4.5. Theorem. *Let $f \in R^n \to R^p$ and $N \hookrightarrow R^p$ a submanifold cut out by independent functions, and of codimension $l \leq p$. Assume that $f \pitchfork N$. Then, $M = f^{-1}(N) \hookrightarrow R^n$ is a submanifold of codimension l (cut out by independent functions).*

Proof. Let f and N be both given at stage A, and assume that $f \pitchfork N$. Let $(A_i \xrightarrow{\gamma_i} A)_{i \in I}$ be a jointly epimorphic family such that for each $i \in I$, $\gamma_i^* N$ is cut by $g_1^i, \dots, g_l^i : R^p \to R$, independent functions in $\mathscr{E}/A_i$. Let $g_1^i = (g_1^i, \dots, g_l^i): R^p \to R^l$. *Claim:* $(g^i \circ \gamma_i^* f)$ is a submersion at every $x \in R^n$ for which $(g^i \circ \gamma_i^* f)(x) = 0$, i.e., at every $x \in R^n$ with $(\gamma_i^* f)(x) \in \gamma_i^* N$. Consider the following commutative diagram in $\mathscr{E}/A_i$, where the first factorization is obtained by applying the chain rule for differentiation, while the second is taking the image of the first map in the first factorization, with $x \in R^n$ arbitrary:

$$\begin{tikzcd}
T_x R^n \arrow[rr, "d(g^i \circ \gamma_i^* f)_x"] \arrow[d, "d(\gamma_i^* f)_x"'] \arrow[dr] & & T_{(g^i \circ \gamma_i^* f)(x)} \\
\mathrm{Im}(d(\gamma_i^* f)_x) \arrow[r, hook] & T_{(\gamma_i^* f)(x)} R^p \arrow[ur, "(dg^i)_{(\gamma_i^* f)(x)}"'] &
\end{tikzcd}$$

By assumption, g^i is a submersion hence $(dg^i)_{(\gamma_i^* f)(x)}$ is locally surjective. The desired conclusion would follow if we could prove that

$$T_{(\gamma_i^*f)(x)}R^p = \mathrm{Im}(d(\gamma_i^*f)_x) + \mathrm{Ker}((dg^i)_{(\gamma_i^*f)(x)}).$$

From $\mathrm{Ker}(g^i) = \gamma_i^*N$ follows easily that

$$\mathrm{Ker}((dg^i)_{(\gamma_i^*f)(x)}) = T_{(\gamma_i^*f)(x)}(\gamma_i^*N)$$

so that the sufficient condition above translates as

$$T_{(\gamma_i^*f)(x)}R^p = \mathrm{Im}(d(\gamma_i^*f)_x) + T_{(\gamma_i^*f)(x)}(\gamma_i^*N)$$

which says, exactly, that $\gamma_i^*f \pitchfork \gamma_i^*N$. By assumption, we only have $f \pitchfork N$, but transversality was defined as a *stable* notion (in the sense of [9], p. 141), hence $\gamma_i^*f \pitchfork \gamma_i^*N$. This establishes the Claim.

Next, by the Preimage theorem (Corollary 2.5), $(g^i \circ \gamma_i^*f)^{-1}\{0\}$ is a submanifold of R^n in $\mathscr{E}/A_i$, of codimension l. Now,

$$\begin{aligned}(g^i \circ \gamma_i^*f)^{-1}\{0\} &= (\gamma_i^*f)^{-1}((g^i)^{-1}\{0\}) = (\gamma_i^*f)^{-1}(\gamma_i^*N)\\ &= \gamma_i^*(f^{-1}(N)) = \gamma_i^*(M).\end{aligned}$$

Therefore, for each $i \in I$, γ_i^*M is a submanifold of R^n in $\mathscr{E}/A_i$ of codimension l. By the Remark 4.4, we have that M is a submanifold of R^n in $\mathscr{E}$, of codimension l. □

The main theorem in the subject of transversality is Thom's transversality theorem (cf. [5], p. 54), of which there are many versions depending on the intended applications. The following version can be proved in our context and has been inspired by one given by Boardman and reproduced in [18], p. 17. But first, we state a result of a general nature.

4.6. Lemma. *Let $\mathscr{E}$ be a topos, with R a commutative ring object of line type in the strong sense. Consider a diagram, where the square in it is a pullback:*

$$\begin{tikzcd} & X \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{P.B.}"] & Y \arrow[d] \\ R^t \arrow[r, "\phi"'] & R^l \arrow[r, "\psi"'] & R^m \end{tikzcd}$$

Assume that Y is a submanifold cut out by independent functions and that ψ is a submersion. Assume also that $\phi \pitchfork X$. Then $\psi \circ \phi \pitchfork Y$.

Proof. By Theorem 4.5, X is a submanifold of R^l. Let $x \in R^t$ be such that $(\psi \circ \phi)(x) \in Y$. Then $\phi(x) \in X$ and since $\phi \pitchfork_x X$ for every $x \in R^t$, we have

$$T_{\phi(x)}R^l = \mathrm{Im}(d_{\phi x}) + T_{\phi(x)}X. \qquad (*)$$

We must show

$$T_{(\psi\circ\phi)(x)}R^m = \mathrm{Im}(d(\psi\circ\phi)_x) + T_{(\psi\circ\phi)(x)}Y. \qquad (**)$$

Let $v \in ((R^m)^D)_{(\psi\circ\phi)(x)}$. Since ψ is a submersion, it follows that there exists (locally) some $u \in (R^l)^D_{\phi(x)}$ such that $\psi\circ u = v$. Apply $(*)$ to u to get, again locally, that there exists $u_1 \in (R^l)^D_x$ as well as $u_2 \in X^D_{\phi(x)}$ such that $u = \phi\circ u_1 + u_2$. Define $v_1 = u_1$ and $v_2 = \psi\circ u_2$. It follows easily that $v = (\psi\circ\phi)v_1 + v_2$ and hence $(**)$ holds. □

Recall from Section 1 the definition of $J^k_h : R^n \to R^{pD_k(n)}$. Notice that if $N \hookrightarrow R^{pD_k(n)}$, the notation $J^k_h \pitchfork N$ is meaningful. We now prove:

4.7. Theorem (Thom's transversality theorem). *Let $p\geq 1$, $k\geq 1$, $n\geq 1$ be given. Let $N\hookrightarrow R^{pD_k(n)}$ be a submanifold cut out by independent functions.*

Then, the following holds:

$$\forall\varepsilon\in\Gamma(R)\ \Big[\varepsilon>0\to\forall h\in\Gamma(R^{p^{R^n}})\ \exists f\in\Gamma(R^{p^{R^n}})$$
$$\Big(f=(f_1,\dots,f_p)\to\bigwedge_{i=1}^{p} f_i\in P^\varepsilon_k(n)\wedge J^k(h+f)\pitchfork N\Big)\Big].$$

Proof. Let $\varepsilon>0$ and $h:R^l\to R^p$ be given. Define $\gamma_h : R^n\times R^{pD_k(n)}\to R^{pD_k(n)}$ by $\gamma_h(x,f) = J^k(h+f)(x)$. *Claim:* γ_h is a submersion. To prove it, it is convenient to identify $f\in\Gamma(R^{pD_k(n)})$ with an s-tuple $(a_{i,\alpha})_{1\leq i\leq p}$, $1\leq\alpha\leq\binom{n+k}{k}$, $\alpha_{i,\alpha}\in\Gamma(R)$. Explicitly,

$$\gamma_h(x,(a_{i,\alpha})) = \left(\frac{h_i^{(\alpha)}(x)+a_{i,\alpha}}{\alpha!}\right)_{i,\alpha}\in R^s.$$

Investigating the Jacobian of γ_h, we see that taking partial derivatives with respect to the $a_{i,\alpha}$ already gives s linearly independent columns. Hence, by Remark 4.2, $\gamma_h\pitchfork N$ and since N is cut out by independent functions, Theorem 4.5 gives that $M=\gamma_h^{-1}(N)$ is a submanifold of $R^n\times R^s$. Consider $\pi : R^n\times R^s\to R^s$, projection onto the second factor. By Remark 3.2, the axiom of density of regular values applies to $\pi\mid_M : M\to R^s$. This says that the following holds:

$$\exists(a_{i,\alpha})\in R^s\ [a_{i,\alpha}\in(-\varepsilon,\varepsilon)\wedge(a_{i,\alpha})\text{ is a regular value of }\pi\mid_M].$$

Let $f_i(x)=\sum_{|\alpha|\leq k} a_{i,\alpha}x^\alpha$, $i=1,\dots,p$. By the choice of the $a_{i,\alpha}$, we have $f_i\in P^\varepsilon_k(n)$. Therefore we have, also

$$\exists f=(f_1,\dots,f_p)\in\Gamma(R^{p^{R^n}})\ \Big[\bigwedge_{i=1}^{p} f_i\in P_k(n)\Big].$$

It remains to show that $J^k(h+f)\pitchfork N$ holds for each such f. Denote by $i_f : R^n\to R^n\times R^{nD_k(n)}$ the map $i_f=\langle\mathrm{id},\ulcorner f\urcorner\rangle$. Notice that for $x\in R^n$, $J^k(h+f)(x)\in N$ if and only if $(\gamma_h \circ i_f)(x) \in N$ and that $J^k(h+f) \pitchfork_x N$ if and only if $(\gamma_h \circ i_f) \pitchfork_x N$. Hence assuming that $(\gamma_j \circ i_f)(x) \in N$, let us prove that $(\gamma_h \circ i_f) \pitchfork_x N$. This will complete the proof.

Apply Lemma 4.6 to the diagram

$$\begin{tikzcd} & M \arrow[r] \arrow[d] & N \arrow[d] \\ R^n \arrow[r, "i_f"'] & R^n \times R^{p^{D_k(n)}} \arrow[r, "\gamma_h"'] & R^{p^{D_k(n)}} \end{tikzcd}$$

Thus, $i_f \pitchfork_x M$ implies $\gamma_h \circ i_f \pitchfork_x N$ and so it is enough to show that $i_f \pitchfork_x M$. But $i_f(x) \in M$ says that $(x, f) \in M$, hence we need to show:

$$T_{(x,f)}(R^n \times R^{n^{D_k(n)}}) = \mathrm{Im}((di_f)_x) + T_{(x,f)}M. \tag{$*$}$$

Since $(-)^D$ preserves finite limits, one can show using this that

$$\begin{tikzcd} T_x R^n \arrow[r, "(di_f)_x"] \arrow[dr, "\langle \mathrm{id}, (d\ulcorner f \urcorner)_x \rangle"'] & T_{(x,f)}(R^n \times R^{n^{D_k(n)}}) \arrow[d, "\cong"] \\ & T_x R^n \times T_f(R^{n^{D_k(n)}}) \end{tikzcd}$$

commutes.

A sufficient condition for $(*)$ is then that

$$T_{(x,f)}M \xrightarrow{(d\pi|_M)_{(x,f)}} T_f(R^{n^{D_k(n)}})$$

be surjective, i.e. that $\pi|_M$ be a submersion at (x, f). Since f was chosen as a regular value of $\pi|_M$, and since $\pi|_M(x, f) = f$ for every $x \in M$, it follows that $\pi|_M$ is a submersion at (x, f) for every $x \in R^n$ such that $(x, f) \in M$, from which the above follows. $\square$

5. Stability and singularities

Let $f \in R^{R^n}$. A *singularity of* f is any $x_0 \in R^n$ such that $j^1_{x_0} f = 0$. A similar definition applies to germs $\phi \in R^{x_0 + \Delta(n)}$, in which case we say also that the germ itself is a singularity.

Let $S^1 \to R^{D(n)} \approx R^{1+n}$ be $\pi^{-1}\{0\}$, where $\pi : R \times R^n \to R^n$ is the projection onto the second factor, hence a submanifold since π is a submersion. Another description of it is: $S^1 = [\![g \in R^{D(n)} \mid \forall x \in D(n)\ g(x) = g(0)]\!]$. Clearly, $x_0 \in R^n$ is a singularity of f if and only if $J^1 f(x_0) = j^1_0(f x_0) \in S^1$.

Let $x_0 \in R^n$ be a singularity of $f \in R^{R^n}$. Call x_0 *non-degenerate* if $J^1 f(x_0) \pitchfork_{x_0} S^1$.

Call $f \in R^n$ a *Morse function* if

$$\forall x \in R^n \ [x \text{ a singularity of } f \to x \text{ is non-degenerate}].$$

An immediate application of Thom's transversality theorem (Theorem 4.7) is the following:

5.1. Corollary. (Density of Morse functions.)

$$\forall \varepsilon \in \Gamma(R) \ [\varepsilon > 0 \to \forall h \in \Gamma(R^{R^n}) \ \exists f \in \Gamma(R^{R^n})$$
$$(f \in P_1^\varepsilon(n) \wedge (h+f) \textit{ is a Morse function})]. \qquad \square$$

5.2. Remark. It is also possible to give a synthetic proof of the following statement (which we leave to the reader):

$$\forall f \in R^{R^n} \ \forall x \in R^n \ [x \text{ is a non-degenerate singularity of } f$$
$$\leftrightarrow x \text{ is a singularity of } f \wedge (\partial^2 f(x)/\partial x_i \partial x_j)_{ij} \# 0].$$

The matrix $(\partial^2 f(x)/\partial x_i x_j)_{ij} \in R^{n^2}$ is called the *Hessian of f (at x)*. However, to proceed to the complete classification of Morse functions in the model $\mathbf{B}^{\mathrm{op}}$, one must either use the reduction to normal form of symmetric matrices from the constructive point of view (as in [16]) or employ the results of Arnold ([1], pp. 46–59). In either case, the result would give a classification of the *unfoldings* of germs of Morse functions (as in [18]).

The *classification of singularities* is with respect to the relation of equivalence of germs, as defined in Section 1. The general problem having proved too difficult, attention was soon focussed on a class of *stable* singularities (or singularities of *stable* mappings). For many pairs of dimensions (n, p), the class of stable smooth mappings $\mathbf{R}^n \to \mathbf{R}^p$ turned out to be open and dense in the Whitney C^∞-topology, as well as easily classifiable according to equivalence (cf. [5]). Stability itself was not a very manageable notion, however; an equivalent condition according to a deep theorem of Mather (cf. [5], Theorem 1.5)), is that of *infinitesimal stability* which is much easier to apply. Moreover, it can be stated and even motivated from the synthetic point of view. In order to understand this motivation, let us recall that a smooth mapping $f: \mathbf{R}^n \to \mathbf{R}^p$ is called *stable* if there exists an open neighborhood W_f of f (in the Whitney C^∞-topology) such that for any $g \in C^\infty(\mathbf{R}^n, \mathbf{R}^p)$, if $g \in W_f$ then $g \sim f$. This says, precisely, that the orbit of f under the action of the group $G = \mathrm{Diff}(\mathbf{R}^n) \times \mathrm{Diff}(\mathbf{R}^p)$, where the action is given by $(g, h) \cdot f = h \circ f \circ g^{-1}$ for $(g, h) \in G$, is open in the Whitney C^∞-topology. It would therefore be enough to have the map

$$G \xrightarrow{\gamma_f} C^\infty(\mathbf{R}^n, \mathbf{R}^p)$$

given by $\gamma_f(g, h) = h \circ f \circ g^{-1}$, a local homeomorphism at $(1_{\mathbf{R}^n}, 1_{\mathbf{R}^p})$. If we had a notion of derivative for maps between function spaces, as well as an appropriate version of the Inverse Function theorem in this generality, a sufficient condition would then be that '$(d\gamma_f)_{(1_{\mathbf{R}^n}, 1_{\mathbf{R}^p})}$' be surjective, i.e., that γ_f be a 'submersion' at $(1_{\mathbf{R}^n}, 1_{\mathbf{R}^p})$. The first requirement is no problem in our context, or for that matter in the context of Frechet manifolds (cf. [5], III. §1); it is the lack of an Inverse Function theorem in both cases which causes Mather's theorem to be non-trivial. However, just as with Frechet manifolds, the above considerations lead us too, in the synthetic approach, to the definition of infinitesimal stability. Let us now justify this remark.

Let $f \in R^{p^{R^n}}$. We can form $T_f(R^{p^{R^n}}) = [(R^{p^{R^n}})^D]_f$, and call this *the object of vector fields along* f. For $1_{R^n} \in R^{n^{R^n}}$, it is easy to see that $s \in T_{1_{R^n}}(R^{n^{R^n}})$ iff $s \in (R^{n^{R^n}})^D$ and $\pi(s) = 1_{R^n}$, i.e., iff s is a *vector field on* R^n (in one of the various forms of the notion afforded by cartesian closedness, (cf. [9], §I.8), namely, as *infinitesimal deformations of the identity map*).

Call $f \in R^{p^{R^n}}$ *infinitesimally stable* if the following holds:

$$\forall w \in \mathrm{Vect}(f)\ \exists s \in \mathrm{Vect}(R^n)\ \exists t \in \mathrm{Vect}(R^p)\ \ (w = df \circ s + t \circ f).$$

Consider next the group

$$G = \mathrm{Units}(R^{n^{R^n}}) \times \mathrm{Units}(R^{p^{R^p}})$$

and let $G \xrightarrow{\gamma_f} R^{p^{R^n}}$ be given by $\gamma_f(g, h) = h \circ f \circ g^{-1}$.

Because R^n is infinitesimally linear, so is $R^{n^{R^n}}$ as well as $\mathrm{Units}(R^{n^{R^n}})$ (cf. [9], Exercise 6.5). Hence, $T_{1_{R^n}}(\mathrm{Units}(R^{n^{R^n}}))$ is an R-module. From [9], Corollary (8.2) follows that if s is a vector field on $R^{n^{R^n}}$ then for every $d \in D$, $s(d)$ is invertible with inverse $s(-d)$. Therefore:

$$T_{1_{R^n}}(\mathrm{Units}(R^{n^{R^n}})) \approx T_{1_{R^n}}(R^{n^{R^n}}) \quad \text{as } R\text{-modules.} \tag{5.3}$$

Since $(-)^D$ preserves finite limits, we also have

$$T_{(1_{R^n}, 1_{R^p})} G \approx T_{1_{R^n}}(R^{n^{R^n}}) \times T_{1_{R^p}}(R^{p^{R^p}}). \tag{5.4}$$

The map γ_f defined above induces a map γ_f^D which restricts (since $\gamma_f(1_{R^n}, 1_{R^p}) = f$) to the linear map

$$T_{(1_{R^n}, 1_{R^p})} G \xrightarrow{(d\gamma_f)_{(1_{R^n}, 1_{R^p})}} T_f(R^{p^{R^n}}),$$

called the *derivative of* γ_f *at* $(1_{R^n}, 1_{R^p})$. The following statement has therefore a meaning in the synthetic context although it does not have one in the usual theory of C^∞-mappings.

5.5. Proposition. *For any* $f \in R^{p^{R^n}}$, *f is infinitesimally stable if and only if γ_f is a submersion at* $(1_{R^n}, 1_{R^p})$.

Proof. Consider the following maps, induced by composition with f:

$$\alpha_f = (R^{n^{R^n}})^D \xrightarrow{(f^{R^n})^D} (R^{p^{R^n}})^D,$$

$$\overline{\alpha_f} = (R^{n^D})^{R^n} \xrightarrow{(f^D)^{R^n}} (R^{p^D})^{R^n};$$

$\overline{\alpha_f}$ is 'composition with df on the left'. Similarly,

$$\beta_f = (R^{p^{R^p}})^D \xrightarrow{(R^{p^f})^D} (R^{p^{R^n}})^D,$$

$$\overline{\beta_f} = (R^{p^D})^{R^p} \xrightarrow{(R^{p^D})^f} (R^{p^D})^{R^n};$$

$\overline{\beta_f}$ is 'composition with f on the right'. One uses $\overline{\alpha_f}$ and $\overline{\beta_f}$ when regarding vector fields as *sections of the projection of the tangent bundle*, which is the usual definition of vector field. The restrictions below are well defined and linear; we denote them by the same letters.

$$T_{1_{R^n}}(R^{n^{R^n}}) \xrightarrow{(\alpha_f)_{1_{R^n}}} T_f(R^{p^{R^n}});$$

$$T_{1_{R^n}}(R^{p^{R^p}}) \xrightarrow{(\beta_f)_{1_{R^p}}} T_f(R^{p^{R^n}}).$$

From the linear isomorphism of (5.3) one derives a new map $(\dot{\alpha}_f)_{1_{R^n}}$ obtained from $(\alpha_f)_{1_{R^n}}$ as follows: if $s \in T_{1_{R^n}}(R^{n^{R^n}}) \approx T_{1_{R^n}}(\mathrm{Units}(R^{n^{R^n}}))$ let $(\dot{\alpha}_f)_{1_{R^n}}(s) = (\alpha_f)_{1_{R^n}}(s^{-1})$.

We claim that the statement we wish to prove will follow from

$$(d\gamma_f)_{(1_{R^n}, 1_{R^p})} = (\dot{\alpha}_f)_{1_{R^n}} \circ \mathrm{proj}_1 + (\beta_f)_{1_{R^p}} \circ \mathrm{proj}_2, \tag{$*$}$$

where $\mathrm{proj}_1, \mathrm{proj}_2$ are the projections associated with the product in (5.4). For, γ_f is a submersion at $(1_{R^n}, 1_{R^p})$ if and only if $(d\gamma_f)_{(1_{R^n}, 1_{R^p})}$ is locally surjective, and on the other hand $(\dot{\alpha}_f)_{1_{R^n}} \circ \mathrm{proj}_1 + (\beta_f)_{1_{R^p}} \circ \mathrm{proj}_2$ is surjective if and only if f is infinitesimally linear by definition. Let us then establish $(*)$. Given $w \in T_f(R^{p^{R^n}})$, $s \in T_{1_{R^n}}(R^{n^{R^n}})$ and $t \in T_{1_{R^p}}(R^{p^{R^p}})$ and $d \in D$, one writes (cf. [9], §I.7)

$$w(d) = df \circ s^{-1}(d) + t(d) \circ f$$

whenever there exists a unique

$$l : D(2) \to R^{p^{R^n}}$$

such that $l(d,0) = df \circ s^{-1}(d)$ and $l(0,d) = t(d) \circ f$ letting then $w(d) = l(d,d)$. It is clear that $l(d_1, d_2) = t(d_2) \circ f \circ (s^{-1}(d_1))$ has these properties so that $w(d) = l(d,d) = t(d) \circ f \circ s^{-1}(d) = (t \circ f \circ s^{-1})(d) = d\gamma_f(s,t)(d)$. This finishes the proof. □

We close this section with some informal remarks about a possible notion of *stability* in this context. In order to define it, some version of the Whitney topology is needed. We propose a notion of open neighborhood of $f \in R^{p^{R^n}}$ in the style of Penon [11, 12], leaving the investigation of its properties for a future occasion.

For $f \in R^{p^{R^n}}$, and $x \in R^n$, $j_x^k f$, the jet of f at x, may be viewed as an element of some R^s, where we have an 'apartness' notion $\#$: invertibility. When we write

$j_x^k f \# 0$ we intend it in this sense.

Write $f \#_\infty 0$ to mean:

$$\exists k \geq 0 \; \exists x \in R^n \; j_x^k f \# 0$$

and $f \#_\infty g$ to mean:

$$(f-g) \#_\infty 0.$$

Notice that $\neg(f \#_\infty 0)$ is the infinitesimal version of the smallness notion employed for polynomials in 4.7 and 5.1 of this paper. But such results need not be true in the infinitesimal versions.

One needs then to introduce what one might call a *Whitney-Penon neighborhood of* f to mean any $U \hookrightarrow R^{p^{R^n}}$ such that

$$\forall g \in R^{p^{R^n}} \; [g \#_\infty f \vee g \in U].$$

A corresponding notion of stability would then read:

$$f \text{ is } \textit{stable} \leftrightarrow_{\mathrm{df}} \forall g \in R^{p^{R^n}} \; [g \#_\infty f \vee g \sim f],$$

where

$$g \sim f \leftrightarrow_{\mathrm{df}} \exists h \in \mathrm{Units}(R^{n^{R^n}}) \; \exists k \in \mathrm{Units}(R^{p^{R^p}}) \; [k \circ f \circ h = g].$$

This definition applies to germs (as well as mappings) "from R^n to R^p".

In this context, Mather's theorem ([5], Thm. 1.5) may be interpreted as saying that an Inverse Function theorem for mappings γ between function spaces, would be true if restricted to the infinitesimally linear γ (recall that the latter can be expressed, synthetically, by the condition "$d\gamma$ is surjective").

6. Unfoldings

As mentioned in Section 5, the notion of stable mapping is important from the point of view of the theory of singularities, on account of the resulting simplification in the classification task. But another motivation for the study of stable maps comes from ideas of R. Thom and his intended applications to the natural sciences (cf. [18], [3] among the sources quoted here). This point of view also led naturally to the consideration of smooth r-parameter families of (potential) singularities, the *unfoldings* of singularities. Which germs gave rise to stable unfoldings was partially answered by means of the notion of a *finitely determined germ*, which is easy to express synthetically:

A germ $\phi \in R^{p^{\neg\neg\{0\}^n}}$ is said to be *k-determined* if

$$\forall \psi \in R^{p^{\neg\neg\{0\}^n}} \; [(j_0^k(\psi) = j_0^k(\phi)) \rightarrow (\psi \sim \phi)]$$

and *finitely determined* if k-determined for some $k \geq 0$. (A weaker version of the above may be more appropriate if trying to work with this notion in this context.)

On the other hand, the categorical point of view introduces an appreciable simplification when dealing with unfoldings. Let us restrict ourselves to germs $\phi : \Delta(n) \to R$ such that $\phi(0) = 0$, in $\mathscr{E}$. For each $r \geq 0$, one has the topos $\mathscr{E}/\Delta(r)$; the functor $\mathscr{E} \xrightarrow{-} \mathscr{E}/\Delta(r)$ is given by:

$$X \mapsto \bar{X} = \left(X \times \Delta(r) \xrightarrow{\pi} \Delta(r)\right), \quad f \mapsto \bar{f} = f \times \mathrm{id},$$

and is logical; so R maps to $\bar{R}$ having the same 'properties' as R. (cf. the remarks of Lawvere in [10] on how to obtain and utilize new models of Synthetic Differential Geometry out of old models, the above method being one of those considered). Denote also by $\mathscr{E}/\Delta(r) \xrightarrow{\partial} \mathscr{E}$ the functor given by

$$Y \to \Delta(r) \mapsto Y; \quad f \mapsto f.$$

By *an r-unfolding of* $\phi : \Delta(n) \to \Delta$ in $\mathscr{E}$ we mean a map $\Phi : \overline{\Delta(n)} \to \bar{\Delta}$ in $\mathscr{E}/\Delta(r)$, satisfying:

$$\begin{tikzcd}
\Delta(n) \arrow[r, "\phi"] \arrow[d, "u^n_{n+r}"'] & \Delta \\
\Delta(n+r) \arrow[r, "\partial_\Phi"'] & \Delta(1+r) \arrow[u, "\pi^{1+r}_1"']
\end{tikzcd}$$

commutes in $\mathscr{E}$.

By virtue of cartesian closedness, we can easily establish the following:

6.1. Proposition. *The following constitute equivalent data for a given* $\phi : \Delta(n) \to \Delta$ *in* $\mathscr{E}$:

(i) *and r-unfolding* Φ *of* ϕ;

(ii) *a map* $f : \Delta(n+r) \to \Delta$ in $\mathscr{E}$ *such that* $f|_{\Delta(n) \times \{0\}} = \phi$ *(usual definition of unfolding, cf.* [18], *Definition* 3.1);

(iii) *a map* $\bar{f} : \Delta(r) \to \Delta^{\Delta(n)}$ *in* $\mathscr{E}$ *such that* $\bar{f}(0) = \phi$. *(This is the point of view of a deformation of* ϕ.)

6.2. Remark. Notice that, because of our definition, an unfolding of a germ in $\mathscr{E}$ "of a map from R^n to R" is *itself* a germ (but in $\mathscr{E}/\Delta(r)$) "of a map from $\bar{R}^n$ to $\bar{R}$". This means that all the definitions that we have given for germs, apply to unfoldings as well. It is also true that theorems about germs (if internally valid) remain true when interpreted as theorems about unfoldings of germs. This is, potentially, a powerful method, not exploited in this paper except for the following simplification of the ordinarily quite complicated notion of *equivalence for unfoldings* (cf. [3], p. 121; [18], p. 55).

Let $\Phi : \overline{\Delta(n)} \to \bar{\Delta}$ and $\Psi : \overline{\Delta(n)} \to \bar{\Delta}$ in $\mathscr{E}/\Delta(r)$ be r-unfoldings (of $\phi : \Delta(n) \to \Delta$ and $\psi : \Delta(n) \to \Delta$). Say that Φ and Ψ are *equivalent r-unfoldings* if there exist invertible germs $\alpha : \overline{\Delta(n)} \to \overline{\Delta(n)}$ and $\beta : \bar{\Delta} \to \bar{\Delta}$ in $\mathscr{E}/\Delta(r)$, such that $\Psi = \beta \circ \Phi \circ \alpha^{-1}$. If, furthermore, $\phi = \psi$, so that both Φ and Ψ are unfoldings of the same germ, ϕ, then we can say that Φ and ψ are *equivalent r-unfoldings of* ϕ if Φ and Ψ are equivalent unfoldings with α and β, and furthermore $\partial\alpha\,|_{\Delta(n)\times\{0\}}$ and $\partial\beta\,|_{\Delta(n)\times\{0\}}$ are both the identity map. (A little bit of work is actually needed to show that this is the same notion as the usual one, but we leave it to the interested reader.)

To see how natural the point of view of comma categories is when dealing with unfoldings, notice that a germ $\phi:\Delta(n)\to\Delta$ in $\mathscr{E}$, when regarded in $\mathscr{E}/\Delta(r)$ via the functor $\mathscr{E}\xrightarrow{-}\mathscr{E}/\Delta(r)$, becomes an r-unfolding of ϕ and is that which is usually labelled the *trivial r-unfolding* of ϕ.

Not always do unfoldings live in the same topos, yet they must be compared. This is done by means of the functors $\mathscr{E}/\Delta(s)\xrightarrow{\gamma^*}\mathscr{E}/\Delta(r)$ induced by composition with maps $\gamma:\Delta(r)\to\Delta(s)$ in $\mathscr{E}$. Thus, let Φ be an r-unfolding and Ψ an s-unfolding (not yet necessarily of the same germ) of germs $\Delta(n)\to\Delta$ in $\mathscr{E}$. A *map* $\Phi\to\Psi$ is a 3-tuple (γ,α,β) with $\gamma:\Delta(r)\to\Delta(s)$, $\alpha:\bar{\Delta}\to\gamma^*\bar{\Delta}$ and $\beta:\gamma^*\overline{\Delta(n)}\to\overline{\Delta(n)}$, such that $\gamma^*\Phi=\alpha\circ\Psi\circ\beta$. If furthermore Φ and Ψ both unfold ϕ, then require that α and β above satisfy

$$\begin{tikzcd} \Delta(n) \arrow[r, "u^n_{n+r}"] \arrow[dr, "u^n_{n+s}"'] & \Delta(n+r) \arrow[d, "\partial_\beta"] \\ & \Delta(n+s) \end{tikzcd} \quad\text{and}\quad \begin{tikzcd} \Delta \arrow[r, "u^1_{1+r}"] \arrow[dr, "u^1_{1+s}"'] & \Delta(1+r) \\ & \Delta(1+s) \arrow[u, "\partial_\alpha"'] \end{tikzcd}$$

are commutative in $\mathscr{E}$. (These diagrams express that α and β are deformations of the identity.)

We end with an elaboration of a remark made by Kock (cf. [8]) about Singularity theory. It is the finitely determined germs which are those for which stable unfoldings exist. Now, the property of being k-determined for a germ at 0, however, is more a property of its k-jet at 0 than of the germ itself; indeed, being k-determined, the germ would 'look like' its jet under a suitable change of coordinates (equivalence). Therefore, what this amounts to, is that it is the k-jets themselves that should be classified under equivalence and, since we are interested in singularities, the task of Singularity theory is then to classify, for each k,n,p, the equivalence classes of 0-preserving maps $D_k(n)\to D_k(p)$ under the relation of equivalence. This is, in fact the point of view which is usually taken when actually giving a classification in low dimensions (cf. [3], Chapter 15; or [18], §5).

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